The fixed point action of the Schwinger model[*][†]

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We compute the fixed point action for the Schwinger model through an expansion in the gauge field. The calculation allows a check of the locality of the action. We test its perfection by computing the 1-loop mass gap at finite spatial volume.

1. INTRODUCTION

The Wilson's Renormalization Group (RG) ensures the existence of perfect actions, i.e. actions which reproduce the continuum independently of the value of the lattice spacing. A feasible objective is the determination of classically perfect actions, which eliminate the cut off effects with restriction to the classical properties of the theory. These actions are related [1] to the fixed point (FP), lying on the critical surface, of a given RG transformation. Although not perfect, the FP actions represent a huge step toward the elimination of the cut off effects [1–3].

The final objective is the application of these ideas to QCD, and steps in this direction have been already done [2,4,5]. The introduction of the interaction between gauge fields and fermions in the context of the FP actions has to cope with several technical problems. Here we study the fermion-gauge field FP interactions in a case much simpler than QCD, the Schwinger model. Since its gauge group is abelian, it is possible to formulate the lattice regularization with non-compact gauge fields, allowing the analytical solution of the pure gauge sector. Therefore, we are able to concentrate the numerical effort in the fermion problem. We test the perfection of the FP action in perturbation theory by computing the 1-loop mass gap in a finite volume, a circle of length L, using the standard action as a "control" action. A non-perturbative approach to the

same problem can be found in the contribution of C.B. Lang and T.K. Pany to this volume [6].

2. THE FP ACTION

A general form of the action of a lattice-regularized abelian gauge theory is, in the non-compact formulation:

$$S = \beta\, S_g(A) + \bar{\psi}\, \Delta(U)\, \psi\,; \qquad (1)$$

the fermion matrix Δ depends on the gauge field through the link variable $U_\mu = \exp(iA_\mu)$. In the classical limit $\beta \to \infty$ the RG transformation assumes the form:

$$S'_g(A') = \min_{\{A\}}\left[S_g(A) + \kappa_g\, \mathcal{K}_g(A', A)\right], \qquad (2)$$

$$S'_F(\bar{\psi}', \psi', U') =$$
$$-\ln \int \left[d\bar{\psi}\, d\psi\right]\exp\left[-\bar{\psi}\,\Delta(U(A^{min}))\,\psi\right.$$
$$\left. - \kappa_F\, \mathcal{K}_F(\bar{\psi}', \psi', \bar{\psi}, \psi, U(A^{min}))\right], \qquad (3)$$

where the primed fields are the degrees of freedom on the coarser lattice, defined through the gauge invariant kernels $\mathcal{K}_g$ and $\mathcal{K}_F$.

In the case of the Schwinger model (and in general for asymptotically free theories) the limit $\beta \to \infty$ is critical. The RG transformation of Eqs. (2) and (3) has a fixed point defined by the FP pure-gauge action S_g^{FP} and fermion matrix Δ^{FP}. If $\mathcal{K}_F$ is quadratic in the fermion fields, the FP action remains a bilinear form of the fermion fields:

$$S^{FP} = \beta\, S_g^{FP}(A) + \bar{\psi}\, \Delta^{FP}(U)\, \psi\,. \qquad (4)$$

We choose the simplest kernel for the gauge field:

$$\mathcal{K}_g(A', A) =$$

$$\sum_{x_B,\mu}\left[A'_\mu(x_B)-\frac{1}{2}(A_\mu(2x_B)+A_\mu(2x_B+\hat{\mu}))\right]^2. \quad (5)$$

The variables x_B label the sites on the coarse lattice in units of its doubled lattice spacing: the site x_B corresponds to the site $2x_B$ in the units of the original lattice. Taking $\kappa_g \to \infty$ we recover the original standard non-compact action as FP action, which is ultralocal, involving only nearest-neighbors interactions.

The fermion kernel has the form:

$$\mathcal{K}_F(\bar{\psi}',\psi',\bar{\psi},\psi,U) =$$

$$\sum_{x_B}[\bar{\psi}'(x_B)-\bar{\Gamma}(x_B,U)][\psi'(x_B)-\Gamma(x_B,U)]. \quad (6)$$

The function $\Gamma(x_B,U)$ defines a gauge-invariant average of the fine fields. Only local, first and second neighbors fine fields contribute to a given coarse field, each with a weight inversely proportional to the total number of coarse fields to which it contributes. Gauge invariance is achieved by parallel-transporting the fine fields to the site x_B through the shortest path.

An additional factor-two renormalization of the gauge field A_μ is required when introducing the interaction, since in $d = 2$ the electric charge e has the dimension of a mass.

We solve the recursive equation (3) in an expansion in the gauge field A_μ. As a result, we find the FP first and second order vertex $R_\mu^{(1)}$ and $R_{\mu\nu}^{(2)}$, defined by (summation over repeated indices is understood):

$$\Delta^{FP}(x,x',U(A)) =$$
$$D^{-1}(x-y) + i\,R_\mu^{(1)}(x-r,y-r)\,A_\mu(r) +$$
$$R_{\mu\nu}^{(2)}(x-r,y-r',r-r')A_\mu(r)\,A_\nu(r') + \ldots \quad (7)$$

$D(x-y)$ stands for the perfect fermion propagator for the free-field case.

The FP couplings decay exponentially with the distance (see Fig. 1), the most important being those coupling fields up to second neighbors. From this observation we conclude that a good parametrization of the full FP action should be
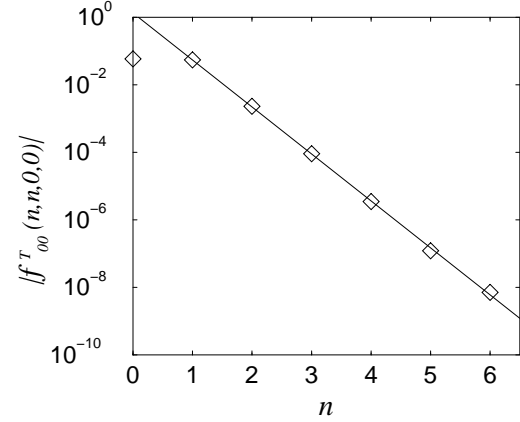
possible by using couplings confined in a 2×2 plaquette.

Figure 1. Exponential decay of $|f_{00}^T(n,n,0,0)|$, defined by $f_{00}^T = \frac{1}{2}\,\mathrm{Tr}(\gamma_0 R_0^{(1)})$; the continuum line is the best fit $\sim \exp(-n/0.44)$.

3. THE MASS GAP

The mass gap of the lattice theory is given by the solution of the equation:

$$|\hat{q}_0|^2 - \Pi_{11}(q_0, q_1 = 0) = 0, \quad (8)$$

where $\hat{q}_\mu = \exp(iq_\mu) - 1$ and $\Pi_{\mu\nu}(q)$ is the lattice vacuum polarization tensor. The perturbative expansion of $\Pi_{\mu\nu}$ defines an analogous expansion for the mass gap; considering the model defined on a cylinder of (spatial) circumference L:

$$m(g, a/L) = g\,m^{(1)}(a/L) + g^3\,m^{(2)}(a/L) + \ldots \quad (9)$$

The scaling limit $a \to 0$, with $g = ea$, gives the continuum mass $m_{ph}^{(c)} = \frac{e}{\sqrt{\pi}}$:

$$\frac{1}{e}\,m_{ph}(ea, a/L) = m^{(1)}(a/L) +$$

$$e^2a^2\,m^{(2)}(a/L) + \ldots \to \frac{m_{ph}^{(c)}}{e} = \frac{1}{\sqrt{\pi}}. \quad (10)$$

We calculated the lowest order term $m^{(1)}(a/L)$, where only the first and second order vertex enter; higher order terms are pure cut off effects. In Fig. 2 we report the results for the Wilson (non-compact) action and the FP action. With the
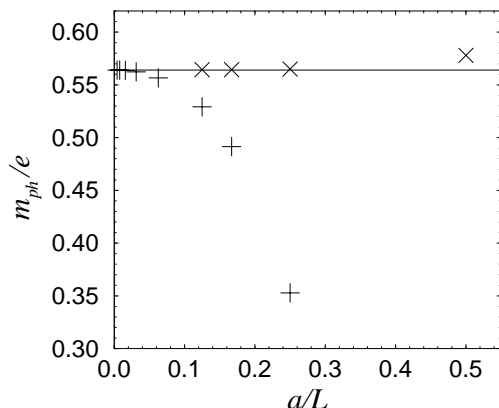

Figure 2. The ratio m_{ph}/e as a function of a/L for the Wilson action (plus) and the FP action (crosses); the solid line is the continuum value $1/\sqrt{\pi}$.

Wilson action we see clear power-like cut off effects, according to the law $m_{ph}/e = 0.5641900 + 1.9 \cdot (a/L)^2 + O((a/L)^4)$; for $L/a = 2$ the lattice theory contains no particle at all. In the case of the FP action, for $L/a > 2$, only tiny ($O(10^{-4})$) deviations from the continuum value are observed (we attribute these deviations to the numerical approximation in the determination of the FP vertices). The deviation for $L/a = 2$ is related to an additional effect [7] exponentially decaying when L/a increases and related to the finite extension of the FP action.

4. 1-LOOP PERFECTION

The recent debate [8] about the 1-loop perfection of the FP action, supported by a formal RG argument, but disproved [8] by an explicit calcu-

lation in the case of the non-linear O(3)-σ model, urged us to consider whether our result can represent or not a check of this property. The property of 1-loop perfection of the FP action is stated in terms of its behavior under RG transformations at finite β. After one RG step:

$$S^{FP} \rightarrow \frac{\beta}{4} \left(S_g^{FP}(A) + \delta S_g(A, \beta) \right) +$$
$$\bar{\psi} \Delta^{FP}(U) \psi + \delta S_F(\bar{\psi}, \psi, U, \beta) ; \qquad (11)$$

1-loop perfection means absence of the lowest-order ($O(1/\beta)$) corrections to the self-reproducing behavior of the FP action under the RG transformation. After the inspection of all possible effective vertices contributing to these lowest-order corrections, we realize that none of them (even if present) would affect the 1-loop mass gap. As a consequence, we conclude that our result represents indeed a check of the *classical* perfection of the FP action of the Schwinger model. A similar situation is found in the framework of the O(3)-σ model.

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